The Frog Problem, limits and bounds in a probabilistic system

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Abstract

In this paper we derive bounds and a limit for the relative speed in a probabilistic system that we call The Frog Problem. The lower bound result is found using a martingale inspired by [1]. Additionally we include some combinatorial remarks about this system.

1 Introduction

Consider the following system where there are K frogs, or nodes, on a lilly pad, i.e a cycle graph. We say that at each iteration each of the K frogs has a probability p of jumping up to an identical lilly pad a unit height above. So for any fixed iteration we can examine at what heights the frogs are positioned. We call two frogs neighbors if on the original cycle graph at height the are neighbors in the graph theoretic sense. We have one additional restriction: After all the frogs have jumped (or not jumped) during a given iteration and a frog is more than one unit above either of its neighbors then it must return to the lilly pad one level below.

In order to define some notion of speed of increase of all the frogs we define $X_{n,k}$ as a random variable for the number of iterations it takes for frog $k \in \{1, ..., K\}$ to jump from level n to level $n+1$ once both of their neighbors are on level n, for $n \in \{1, ..., N\}$.We also define $L_n(i,j)$ as the relationship indicator function, i.e. $L_n(i,j) = 1$ if the node (n,i) and $(n+1,j)$ are connected, and 0 otherwise. Let r be $E\sum_j L_n(i,j)$ for all i.

Next we define the random variable $T(N,K)$the time it takes for all K frogs to jump past the N^{th} level i.e.,

$$T(N,K) \stackrel{\text{def}}{=} \max_{k_1,k_2,...,k_N} \{(\prod_{n=2}^{N} L_{n-1}(k_{n-1},k_n)) \sum_{n=1}^{N} X_{n,k_n}\}.$$

We want to introduce some measure of asymptotic speed. Thus we define $S(K) \stackrel{def}{=} \lim_{N\to\infty} \frac{N}{T(N,K)}$. Our goal in this paper is to find bounds and $\lim_{K\to\infty} S(K)$. Additionally we include some bounds on slightly modified systems and combinatorial results.

2 Finite K and Markov Chains

We can theoretically construct for finite K a Markov chain process that gives us the asymptotics about the time spent in each state, i.e. the stationary distribution. Standard references can be found in Chapter 11 of Grinstead's book [2].

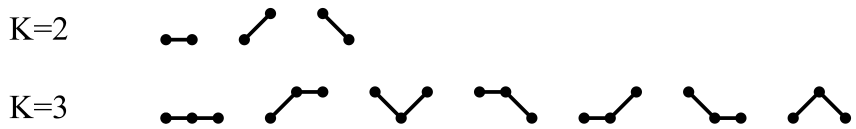

Figure 2.1: A visualization of the different states given two or three frogs

$$M = \begin{bmatrix} Q & \vdots & R \\ \cdots & & \cdots \\ 0 & \vdots & I_K \end{bmatrix}$$

For $K = 2$ we have

$$Q = \begin{bmatrix} q^2 & pq & pq \\ 0 & pq + q^2 & 0 \\ 0 & 0 & pq + q^2 \end{bmatrix} \qquad \text{and} \qquad R = \begin{bmatrix} p^2 & 0 & 0 \\ pq & p^2 & 0 \\ pq & 0 & p^2 \end{bmatrix}.$$

It can be easily checked that

$$[I - Q]^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2p-3}{p(p-2)} \\ \frac{1}{2p(1-p)} \\ \frac{1}{2p(1-p)} \end{bmatrix}$$

The existence of the speed $S(K)$ comes directly from the fact that any absorbing Markov chain will be absorbed in finite time. This leads to our idea of finding some bounds and convergence properties of S(K) as K goes to infinity. The main findings are presented in the next section.

3 Main Theorem

In this section, we explore the bounds and the limits in a slightly different problem. In this case, the frogs cannot jump before the neighbors reach the same level, i.e. we eliminate the situations when all of the frogs on a chain jump at once. This problem was first approached by Chang and Nelson. The proof of our theorem also inherits their technique of martingale [1].

[1]references about Martingale can be found in [3]

Theorem 3.1 *If the moment generating function of the time it takes for a frog to jump exists for some positive* θ_0 [2]

$$\phi(\theta) \stackrel{\text{def}}{=} E[e^{\theta X_{n,k}}] < \infty \quad \text{for } 0 < \theta \leq \theta_0$$

then the asymptotic speed for all of the frogs to jump higher than some fixed level $S(K)$ *is bounded below* $\frac{1}{t^*}$*, where*

and

$$t^* = \inf\{t \geq 1 | rm(t) < 1\}$$

and

$$m(t) = \inf_{0<\theta<\theta_0} \{e^{-\theta t}\phi(\theta)\}.$$

Lemma 3.2 *Under the assumption Theorem 3.1, let* $\mathcal{F}_n$ *be the minimal* σ*-algebra generated by* $\{X_{n,k}, k = 1, ..., K, m = 1, ..., n\}$*. Let*

$$M_n(\theta) = \frac{1}{(r\phi(\theta))^n} \sum_{k_1=1}^{K} \sum_{k_2=1}^{K} ... \sum_{k_N=1}^{K} \left(\prod_{m=2}^{n} L_{m-1}(k_{m-1}, k_m) \right) e^{\theta \sum_{m=1}^{n} X_{m,k_m}}$$

Then $\{M_n(\theta), \mathcal{F}_n, n \geq 1\}$ *is a nonnegative martingale for* $\theta \leq \theta_0$*.*

Proof. We show that $E[M_{n+1}(\theta)|\mathcal{F}_n] = M_n(\theta)$. First observe that

$$E[M_{n+1}(\theta)|\mathcal{F}_n] = \frac{1}{(r\phi(\theta))^n} \sum_{k_1=1}^{K} \sum_{k_2=1}^{K} ... \sum_{k_N=1}^{K} E\left(\sum_{k_{n+1}=1}^{K} (\prod_{m=2}^{n+1} L_{m-1}(k_{m-1}, k_m)) e^{\theta \sum_{m=1}^{n+1} X_{m,k_m}} | \mathcal{F}_n \right)$$

Since $L_{m-1}(i,j)$ and for $m \leq n$, are $\mathcal{F}_n$-measureable,

$$E\left(\sum_{k_{n+1}=1}^{K} (\prod_{m=2}^{n+1} L_{m-1}(k_{m-1}, k_m)) e^{\theta \sum_{m=1}^{n+1} X_{m,k_m}} | \mathcal{F}_n \right)$$
$$= (\prod_{m=2}^{n} L_{m-1}(k_{m-1}, k_m)) e^{\theta \sum_{m=1}^{n} X_{m,k_m}}) E\left(\sum_{k_{n+1}=1}^{K} L_n(k_n, k_{n+1}) e^{\theta X_{n+1,k_{n+1}}} | \mathcal{F}_n \right)$$

Because the random variables are iid,

$$E\left(\sum_{k_{n+1}=1}^{K} L_n(k_n, k_{n+1}) e^{\theta X_{n+1,k_{n+1}}} | \mathcal{F}_n \right) = \sum_{k_{n+1}=1}^{K} E(L_n(k_n, k_{n+1}) E(e^{\theta X_{n+1,k_{n+1}}})$$
$$= r\phi(\theta)$$

Simple substitution completes the proof. ∎

[2]in the case where $X_{i,j}$'s are geometric, $\theta_0 = -\ln(q)$

Lemma 3.3 *For the system as previously defined,*

$$\limsup_{N\to\infty} \frac{T(N,K)}{N} \leq t^*, \quad a.s.$$

Proof. It follows from our definition of $T(N,K)$ that,

$$e^{\theta T(N,K)} = \max_{k_1,\ldots,k_N} \left\{ \prod_{n=2}^{N} L_n(k_{n-1},k_n) e^{\theta \sum_{n=1}^{N} X_{n,k_n}} \right\} \leq (r\phi(\theta))^N M_N(\theta).$$

Since $M_n(\theta)$ is a martingale and $E[M_1(\theta)] = K$, we have the following inequality,

$$E\left[e^{\theta T(N,K)}\right] \leq \frac{K}{r}(r\phi(\theta))^N.$$

Next, using the above inequality and Markov's inequality yields

$$\begin{aligned} P\left(\frac{T(N,K)}{N} > t\right) &= P\left(e^{\theta T(N,K)} > e^{\theta N t}\right) \\ &\leq e^{-\theta N t} E\left[e^{\theta T(N,K)}\right] \\ &\leq \frac{K}{r}(e^{-\theta t} r\phi(\theta))^N \end{aligned}$$

Thus, $\sum_N P\left(\frac{T(N,K)}{N} > t\right) < \infty$ if $(e^{-\theta t} r\phi(\theta)) < 1$, i.e. $rm(t) < 1$. By Borel-Cantelli lemma, we have

$$P\left(\limsup_{N\to\infty}\left(\frac{T(N,K)}{N} > t\right)\right) = 0$$

which implies

$$\limsup_{n\to\infty} \frac{T(N,K)}{N} \leq t^*, \text{a.s}$$

∎

Proposition 3.4 *In our specific case where $X_{i,j}$ are approximately geometric,*

$$\phi(\theta) = \frac{pe^\theta}{1-qe^\theta}$$

Proof.

$$\phi(\theta) = E[e^{\theta X_{n,k}}] = \sum_{n=1}^{\infty} pq^{n-1}e^{\theta n} = \frac{p}{q}\sum_{n=1}^{\infty}(qe^\theta)^n = \frac{pe^\theta}{1-qe^\theta}$$

∎

Theorem 3.5 *Given K^2 iterations, the probability that all frogs have passed level N, where $N = aK^2$, $a < \frac{-lnq}{lnr}$, equals $1-\alpha\beta^{K^2}$ where $\beta < 1$. That is,*

$$P(T(aK^2,K) \leq K^2) \overset{K\to\infty}{\longrightarrow} 1$$

Proof.

$$\begin{aligned} P(T(N,K) > K^2) &= P(e^{\theta T(N,K)} > e^{\theta K^2}) \\ &\leq e^{-\theta K^2} E[e^{\theta T(N,K)}] \\ &\leq e^{-\theta K^2} \frac{K}{r} (r\phi(\theta))^N \\ &= e^{-\theta K^2} \frac{K}{r} \left(\frac{rpe^\theta}{1-qe^\theta} \right)^N \\ &= \frac{K}{r} \left(\frac{\left(\frac{rpe^\theta}{1-qe^\theta} \right)^a}{e^\theta} \right)^{K^2} \end{aligned}$$

Since $\left(\frac{rpe^\theta}{1-qe^\theta} \right) < r$ and note that $r = 3$ in this case, we have

$$\left(\frac{\left(\frac{rpe^\theta}{1-qe^\theta} \right)^a}{e^\theta} \right) < 1,$$

the expression goes to 0 exponentially. This completes the proof. ■

Corollary 3.6 *The number of steps it takes to get to the aK^2 level in the frog case is also $\leq K^2$ a.s.*

Proof. The proof comes from the observation that in the speed of the system we describe in the current section is less than that of our initial system. ■

4 Different types of lilly pads

4.1 Segment case

Here we are going to assume that the frogs sit on a line segment instead of on a circle, i.e. the first and last frogs are not connected.

Lemma 4.1 *If the speed $S(K)$ is still defined as above, then $S(K) \geq S(K+1)$*

Proof. The proof can be obtained by simple reasoning. When we have K+1 frogs, the speed of the first K frogs is the same as $S(K)$, so adding one more frog at the end cannot increase the speed of the system. Therefore $S(K) \geq S(K+1)$ ■

Corollary 4.2 $\lim_{K\to\infty} S(K)$ *exists and the limit is greater than $1/t^*$, where t^* is the value obtained in Theorem 3.1*

4.2 No restriction

In this case, the frogs have no restrictions and can jump independently

Proposition 4.3 $T(N,K) = \max_{i=1,\ldots,K}\{\sum_{j=1}^{N} X_{j,i}\}$

4.3 Maximum restriction

In this case, each frogs have to wait for all frogs to reach the same level to start jumping.

Proposition 4.4 *Given that we start at a state with n frogs on the bottom level, then the expected iteration for the lowest level to increase is*

$$\sum_{i=1}^{n} \binom{n}{i} (-1)^{i+1} \frac{1}{1-q^i}$$

Proof. Since the frogs on the bottom level do not have to wait for their neighbors, the time it takes a bottom frog to go up one level is geometrically distributed. Then the expected time equals the expectation of n geometric random variables. Let $Y = \max\{X_1, ..., X_n\}$. Then

$$\begin{aligned} E[Y] &= \sum_{k=0}^{\infty} P(Y > k) = \sum_{k=0}^{\infty} (\sum_{i=0}^{n} \binom{n}{i} (-1)^{i+1} q^{ki}) \\ &= \sum_{i=1}^{n} \binom{n}{i} (-1)^{i+1} \sum_{k=0}^{\infty} q^{ki} = \sum_{i=1}^{n} \binom{n}{i} (-1)^{i+1} \frac{1}{1-q^i} \end{aligned}$$

■

Remark: Since the geometric distribution can be approximated by the exponential distribution, the expected time can be obtained by finding the mean of the maximum of n exponential random variables. The result is well-known,

$$E[\max\{X_1, ..., X_n\}] = \sum_{i=1}^{n} \frac{1}{i}$$

Theorem 4.5 *The expected time to get to level N is $N \sum_{i=1}^{K} \binom{K}{i} (-1)^{i+1} \frac{1}{1-q^i}$*

Proof.

$$E[T(N,K)] = E[\sum_{i=1}^{n} \max\{X_{i,1}, ..., X_{i,K}\}] = \sum_{i=1}^{n} E[\max\{X_{i,1}, ..., X_{i,K}\}] = N \sum_{i=1}^{K} \binom{K}{i} (-1)^{i+1} \frac{1}{1-q^i}$$

■

5 Combinatorial results

Proposition 5.1 *The number of possible states of K frogs on a directed cycle are enumerated by the central trinomial coefficients, i.e. number of states $= \binom{K+1}{0}_2$*

Proof. Upward, downward and horizontal steps increase the current level by 1, -1 and 0 respectively. Then the number of states is equal to the number of combinations of $x_i's$, where $x_i \in \{1, 0, -1\}$, such that $\sum_{i=1}^{K+1} x_i = 0$. That is equal to the free coefficient of x in the expansion of $(1 + x + x^{-1})^{K+1}$, which is the $(K+1)^{th}$ central trinomial coefficient. ■

Proposition 5.2 *If we count the number of states by start counting from a frog on the lowest level, the number of such states is the* $(K+1)^{th}$ *Motzkin number.*

Proof. The number of such states is the number of lattice paths that go from $(0,0)$ to $(0,K+1)$ without crossing the x-axis. This is exactly the definition of the Motzkin number. ■

6 Future Research

One possible goal for future research is to find a similar bound on the time that the system first reachs level N. The current problem is that we cannot construct this value the same way we did with $T(N,K)$. It possibly is the minimum of a sum of the $X_{i,j}$'s, and if we can find the exact formula for it we can come up with a similar bound by using $\phi(\theta)$ where $\theta < 0$.

References

[1] C. S. Chang and R. Nelson. Bounds on the speedup and efficiency of partial synchronization in parallel processing systems. *J. ACM*, 42(1):204–231, January 1995.

[2] Charles M. Grinstead and J. Laurie Snell. *Introduction to Probability.* Student Mathematical Library. American Mathematical Society, Providence, RI, 1997.

[3] S.R.S. Varadhan. *Probability Theory.* American Mathematical Soc.